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GROUP DIVISIBLE DESIGNS THROUGH HADAMARD MATRIX

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ABSTRACT

In this paper, we have developed some methods of construction of group divisible designs using Hadamard matrix of size p. Method of construction is supported by suitable example. Here, the constructed group divisible designs are happened to be semi Regular group divisible designs.

KEYWORDS: *Hadamard Matrix, Characteristic Roots and Group Divisible Designs*

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1. INTRODUCTION

Yates (1936) introduced the Balanced Incomplete Block Design for agriculture experiments. But, the drawback of the Balanced Incomplete Block Design is that it is not available for all possible number of treatments because of its parametric relations: (i) b k = vr, (ii) r (k - 1) = (v - 1) and (iii) b ≥ v.

To overcome this problem, Yates (1936) introduced another class of design which he called Lattice Design, which is available for all possible number of treatments, however, the treatment must be either perfect square or cubic. This type of incomplete block design could not satisfy the requirements of the agriculture experiments. Hence, Bose and Nair (1939) introduced another class of incomplete block design which they called Partially Balanced Incomplete Block Design (PBIBD). The advantage is that it is available for all number of treatments, even having a small number of replications. Several authors developed the method of construction and properties of PBIBD. Again, Bose and Shimamoto (1952) classified the PBIBD with two associated class into five categories.

- Group Divisible design
- Simple Partially Balanced Incomplete Block Design
- Triangular type Partially Balanced Incomplete Block Design
- Latin Square type Partially Balanced Incomplete Block Design and
- Cyclic Partially Balanced Incomplete Block Design.

Among these, the simplest and perhaps most important class of design is Group Divisible Design. A Group Divisible Design is a two associate class of PBIBD for which the treatment may be divided into m – groups of n – distinct treatments each such that any two treatments that belong to the same group are called first associates and any two treatments that belong to the different groups are second associates. The association scheme can be exhibited by placing the treatments in an n x m rectangle, where the columns form the groups.

Further, Bose and Conner (1952) have made a careful study of combinatorial properties of Group Divisible Design and classified them into three sub types depending upon the quantities (i) Q = r – λ1 and (ii) P = r k – v λ2 .

The Group Divisible Design is classified into three classes from the consideration of the characteristic roots of NN’ matrix as follows:

(a) Singular, if , r – λ1 = 0. Otherwise, Non singular, if r – λ1 > 0.

Further, non singular group divisible designs are classified as (b) Semi – Regular, if $r\ k - v\ \lambda_2 = 0$, and (c) Regular, if $r\ k - v\ \lambda_2 > 0$.

As we know from Bose and Conner (1952) that the existence of a Balanced Incomplete Block Design implies the existence of Singular Group Divisible Design. Considering this idea, we have developed the construction of Singular Group Divisible Designs with repeated blocks (SGD-RB) from the Balanced Incomplete Block Designs with repeated blocks.

It is further seen that the Singular Group Divisible Designs obtained here are having the same parameters, as discussed by Clatworthy (1973). However, the Singular Group Divisible Designs discussed here gives the isomorphic solutions to the solutions given in Clatworthy (1973). Hence, the Singular Group Divisible Designs developed here can be claimed as the new solutions of Singular Group Divisible Designs. To show the isomorphism of the new solutions with existing one, the triplets of the set of treatments in all blocks are prepared and frequency of the number of triplets repeated in the design is counted and its frequency distribution is compared with the corresponding frequency distribution of frequency of the number of triplets of the existing designs. Ghosh and Das (1989) discussed the construction of two way group divisible designs. Further, Ghosh and Bhimani(1990) carried out different method of construction of group divisible designs. Later, Ghosh and Das (1993) carried out the construction of group divisible designs with partial balance for group comparisons. Very recently, Ghosh and Sinojia (2020) developed the construction of group divisible designs using Kronekker product of design.

In this investigation, we have considered Hadamard matrices of size p. Such that HpHp’ = pIp. Next, delete the first column of this Hadamard matrix of size p. so the dimension of the remaining matrices becomes p (p - 1), called this matrix A. Next code these elements 1 of matrix A as 1 while elements -1 as 2, ands called this matrix B. By keeping 1st column of matrix B as such and then adding 2, 4, 6,…, 2(n-1) with every dements of 2nd , 3rd , 4th ,…, (n-2)th columns respectively of matrix B. We obtained a partially Balance Incomplete Block design with parameters v = 2(p-1), b = p, r = p/2, k = p-1, λ1= 0 , n1 = 1, λ2 = p/4, n2 = 2(p-2), n = 2, m = (p – 1) along with the associate matrix as $P_{ij}^{1} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,(m-1) \end{pmatrix}$ and $P_{ij}^{2} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,(m-2) \end{pmatrix}$.

Here, our effort is to construct partially balanced incomplete block design through Hadamard matrix. Hence, we defined the Hadamard matrix in section 2.

2. DEFINITION OF HADAMARD MATRIX

A matrix Hn of order n is said to be Hadamard matrix if HnHn´ = Hn´Hn= nIn, where n is multiple of four with

$H_2 = \begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix}.$

3. METHOD OF CONSTRUCTION

Consider a Hadamard Matrix of order p such that $H_pH_p' = p\ I_P$. Next, delete the first column of this Hadamard matrix of size p so the dimension of the remaining matrix becomes p x (p-1). Call this matrix by A. Next code the element 1 of matrix A as 1 while element -1 as 2 and call this matrix by B.

3.1 Group Divisible Design with (p - 1) Treatments

Theorem 3.1: By keeping first column of matrix B as such and then adding 2, 4, 6,..., 2(p -1) with every element in 2^{nd}, 3^{rd}, 4^{th} ..., $(n-2)^{th}$ columns respectively of matrix B, we obtain Group divisible partially balanced incomplete block design with parameters v = 2(p - 1), b = p, r = p/2, k = p - 1, $\lambda_1 = 0$, $n_1 = 1$, $\lambda_2 = p/4$, $n_2 = 2(p - 2)$, n = 2, m = (p – 1) along with the association matrix.

$$P_{ij}^{1} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,(m-1) \end{pmatrix} \quad \text{and} \quad P_{ij}^{2} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,(m-2) \end{pmatrix}$$

Proof: Let Hp be the hadamard matrix of size p, whose elements are +1 and -1. Now, delete the first column of this Hadamard matrix. Let us call, this matrix as A. However, the dimension of this matrix reduces to p x (p -1). Next, we replace the element +1 of matrix A by 1 and element -1 by 2. This matrix is called B. First column of matrix B is kept as such, while element in 2^{nd}, 3^{rd}, $(n-2)^{th}$ columns of matrix B is added by 2, 4, 6,..., 2(n – 1) respectively. Call this matrix by D. In first row, all the elements are odd. It is noted here that the difference of elements between any two consecutive columns of first row is always two. In design matrix B, we have added 2, 4, 6, ... to get design D, that is, each subsequent number is multiple of 2, so the number of treatment for resulting design is 2(p -1). Now in design matrix D, we have (p -1) columns. Since in design D, number of rows is same as in matrix B and matrix A and hence number of blocks for resulting design is p. In hadamard matrix of size p, each column contains p/2 time +1 and p/2 times -1. Further +1 is replaced by 1 and -1 by 2 and hence for resulting Group Divisible design, each treatment is repeated p/2 times. Moreover, it is observed that treatment i and i+1 do not occur together in any block and hence $\lambda_1 = 0$ and such treatments is always 1 and hence $n_1 = 1$, where treatment i represent odd number of treatment. For an example, if i = 1 then i + 1 = 2 and hence for treatment 1 first associate treatment is 2 and vice versa. Next i will be 3. That is, first associate treatment for 1, 3, 5, 7, 9,…., (2p - 3) are always 2, 4, 6, 8,….,2(p - 1) and vice versa. Further with treatment i, the remaining 2(p -2) treatment will occur p/4 times only, and hence the number of second associates treatments are 2(p – 2), that is, $n_2 = 2$ (p-2) and $\lambda_2 = p/4$. The design D satisfied all the primary and secondary parametric relation of PBIB Design and hence the resulting design is partially balanced incomplete block design of two associate classes. Further, we observed that n = 2 and $n_1 = 1$, again $m = \frac{v}{n} = \frac{2(p-1)}{2} = (p-1)$, hence m and n are positive integers, so this PBIB design satisfies the criteria of group divisible design. Again $(r-\lambda_1) = \frac{P}{2} - 0$ which is always greater than zero and is integer as p is always multiple of 2 and greater than and equal to 4. Moreover, $(rk - v\lambda_2) = \frac{P}{2}(p-1) - 2(p-1)x\ p/4$ is equal to zero. Since $(r - \lambda_1)$ and $(rk - v\lambda_2)$ satisfied the characteristics roots of NN' matrix and hence the resulting PBIB design is semi regular group divisible design along with association matrices.

$$P_{ij}^1 = \begin{pmatrix} 0 & 0 \\ 0 & n(m-1) \end{pmatrix} \text{ and } P_{ij}^2 = \begin{pmatrix} 0 & 0 \\ 0 & n(m-2) \end{pmatrix}.$$

Example 3.1: Construct a semi regular group divisible design with parameters v = 14, b = 8, r = 4, k = 7, $\lambda_1 = 0$, $n_1 = 1$, $\lambda_2 = 2$, $n_2 = 12$, n = 2 and m = 7.

Consider a Hadamard matrix of size p = 8. This is given below.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

After deleting first column and replacing 1 by 1 and -1 by 2, we get another matrix B with dimension 8 X 7. This is given below:

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}$$

Next, we kept first column as such and added 2, 4, 6, 8, 10, 12 with all the elements of columns 2nd, 3rd, 4th, 5th, 6th and 7th respectively. The resulting design D is given by

$$D = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 2 & 3 & 6 & 7 & 10 & 11 & 14 \\ 1 & 4 & 6 & 7 & 9 & 12 & 14 \\ 2 & 4 & 5 & 7 & 10 & 12 & 13 \\ 1 & 3 & 5 & 8 & 10 & 12 & 14 \\ 2 & 3 & 6 & 8 & 9 & 12 & 13 \\ 1 & 4 & 6 & 8 & 10 & 11 & 13 \\ 2 & 4 & 5 & 8 & 9 & 11 & 14 \end{bmatrix}$$

By considering rows as blocks, D gives semi regular group divisible design with parameters v = 14, b = 8, r = 4, k = 7, $\lambda_1 = 0$, $n_1 = 1$, $\lambda_2 = 2$, $n_2 = 12$, n = 2, m = 7 along with association matrices as

$$P_{ij}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 12 \end{pmatrix}, \text{ and } P_{ij}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix},$$

However, this is not a new semi-regular group divisible design as it is reported as SR 80 in Clatworthy (1973). However, the method of construction is different.

3.2: Construction of Group Divisible Design with v = p Treatments

Using matrix B from section 5.3.1, we can again construct another series of group divisible design with parameters v = p, b = p, $r = \frac{p}{2} = \frac{k}{2}$, $\lambda_1 = 0$, $n_1 = 1$, $\lambda_2 = \frac{p}{4}$, $n_2 = p - 2$, n = 2 and $m = \frac{p}{2}$. This is shown in theorem 3.2.

Theorem 3.2: Let us construct a matrix B using section 5.3.1. Let us keep first $\frac{P}{2}$ columns of design Matrix B as such and delete the remaining ($\frac{P}{2}$ -1) columns. Adding 2, 4, 6, …2($\frac{p}{2}$ - 1) with 2nd , 3rd , 4th … $\frac{p}{2}$ column of matrix B there always exits a group divisible design with parameters v = p = b, r = k = $\frac{P}{2}$, λ_1= 0, n_1=1, λ_2 = $\frac{P}{4}$, n_2 = p - 2, n = 2 and m = $\frac{P}{2}$.

Proof: First of all, construct a design matrix B using section 3.1. Now, we are keeping the first $\frac{P}{2}$ columns intact and delete the remaining ($\frac{P}{2}$ - 1) columns from design B. Since with design matrix B, we have added 2, 4, 6, 8, …, 2($\frac{p}{2}$ - 1) with the elements of 2nd , 3rd ,4th,…, ($\frac{p}{2}$)th columns. The elements present in incidence matrix B are 1 and 2 so number of treatments for resulting design is 2 x $\frac{P}{2}$ = p. Obviously, number of blocks is p and r = $\frac{P}{2}$. Since $\frac{P}{2}$ columns retains only new design and hence k =$\frac{P}{2}$. As per section 3.1, the remaining parameters are λ_1 = 0, n_1 = 1, λ_2 = $\frac{P}{4}$, n_2 = p - 2. The resulting group divisible design happened to be symmetric semi-regular group divisible design.

Example 3.2: Construct a symmetric semi-regular group divisible design with parameters v = b = 8, r = k = 4, λ_1 = 0, n_1 =1, λ_2 = 2, n_2 = 6, n = 2, and m = 4.

Let us construct a Hadamard matrix of size 8 as p = v = 8 which is given as

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Next, after deleting first column and decoding +1 as 1 and -1 as 2 we get another matrix of dimension 8 x 7 provided first row remain as such. Denote this matrix by B. This is given below:

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}$$

Delete the last three column from matrix B. Keeping elements in first column as such and add 2, 4, 6, with 2nd, 3rd, 4th column of matrix B. Finally, we obtain matrix D as given below:

$$D = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 4 & 6 & 7 \\ 2 & 4 & 5 & 7 \\ 1 & 3 & 5 & 8 \\ 2 & 3 & 6 & 8 \\ 1 & 4 & 6 & 8 \\ 2 & 4 & 5 & 8 \end{bmatrix}$$

The resulting design D is a partially balanced incomplete block design by considering rows as number of block. Blocks of this design are given below.

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 \\ 5 & 6 & 6 & 5 & 5 & 6 & 6 & 5 \\ 7 & 7 & 7 & 7 & 8 & 8 & 8 & 8 \end{bmatrix}$$

This PBIB design is happened to be the group divisible design. Here, the parameters of group divisible design are $v = 8 = b$, $r = 4 = k$, $\lambda_1 = 0$, $n_1 = 1$, $\lambda_2 = 2$, $n_2 = 6$, $n = 2$, $m = 4$. Further, the characteristics roots of group divisible design satisfied $r - \lambda_1 = 4 – 0 > 0$ and $rk - v\lambda_2 = 4\times4 - 8\times2 = 0$ and hence the resulting group divisible design is a symmetric semi regular group divisible (SRGD) design.

This design is not a new SRGD as it is reported as SR36 in Clatworthy (1973). However, it is an alternate method to obtain SR36 design.

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